TOPIC: POWER SERIES & TAYLOR SERIES

Intro to Power Series

◆ A power series is an infinite series that also involves a variable, ____, & can be thought of as an "infinite ____________."

► Power series have ____________ determined by c_n and _________ at $x = a$.

EXAMPLE For $c_n = \frac{1}{n!}$, write the first 4 terms of the $(\boldsymbol{A})$ infinite series & $(\boldsymbol{B})$ power series centered at $a = 2$.

Recall: Infinite Series	New: Power Series
$\sum_{n=0}^{\infty} c_n = c_0 + c_1 + c_2 + \cdots$	$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$
$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{\ \ !} + \frac{1}{\ \ !} + \frac{1}{\ \ !} + \frac{1}{\ \ !} + \cdots$	
$= \quad + \quad + \quad + \quad + \cdots$	

◆ We can determine convergence of, perform operations on, and represent functions with power series.

EXAMPLE Determine where the power series are centered and list the four first terms of the given series.

$(\boldsymbol{A})$

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1) \cdot 3^n}$$

$(\boldsymbol{B})$

$$\sum_{n=0}^{\infty} (-1)^n n!\, x^n$$

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Radius of Convergence

◆ A power series only converges for certain values of x which is an interval determined by $|x-a| < R$.

► The radius of convergence ___ tells us how far away from the center ___ that the series converges.

EXAMPLE Find the radius of convergence for the following series.

$$\sum_{n=0}^{\infty} \frac{n \cdot (x-2)^n}{3^n}$$

HOW TO: Find Radius of Convergence

1) Apply **convergence** test

Ratio: $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| < 1$

Root: $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} < 1$

Geometric: $\sum ar^n; |r| < 1$

2) Put in $|x-a| < R$ form to find R

R can be:

- ► a __________ value
- ► ______ if the limit = ______
- ► ______ if the limit = ______

EXAMPLE Find the radius of convergence of the series.

$(\boldsymbol{A})$

$$\sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

$(\boldsymbol{B})$

$$\sum_{n=0}^{\infty} n^n x^n$$

$(\boldsymbol{C})$

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n4^n}$$

HOW TO: Find Radius of Convergence

1) Apply **convergence** test

Ratio: $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| < 1$

Root: $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} < 1$

Geometric: $\sum ar^n; |r| < 1$

2) Put in $|x-a| < R$ form to find R

R can be:

- a **finite** value
- $\mathbf{0}$ if the limit = ∞
- ∞ if the limit = $\mathbf{0}$

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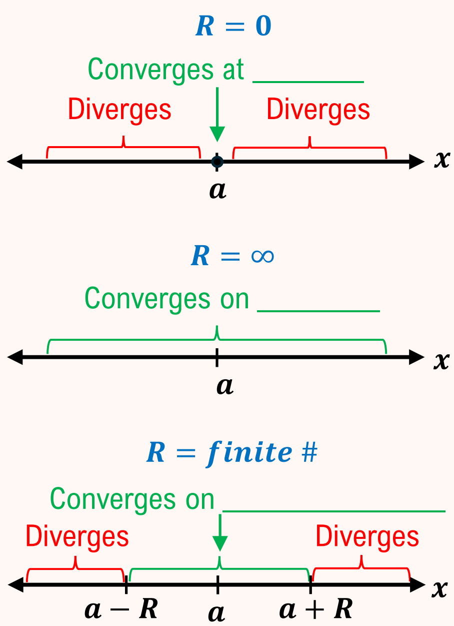
Interval of Convergence

◆ Recall: Radius of convergence $\boldsymbol{R}$ is how far from center $\boldsymbol{a}$ that a power series converges.

► The *interval* of convergence is found by ____________ the inequality $|x-a| < R$.

EXAMPLE Find the interval of convergence of the power series below given its radius of convergence.

New **Interval of Convergence**



$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n4^n}$$

Radius of Convergence: $R = 4$

Inequality: $|x+3| < 4$

◆ We must also determine convergence of ____________ by plugging them in for x into the original power series.

EXAMPLE Determine endpoint convergence for the power series.

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n4^n}$$

$x =$ ___: $\displaystyle\sum_{n=1}^{\infty} \frac{(\quad +3)^n}{n4^n}$

$x =$ ___: $\displaystyle\sum_{n=1}^{\infty} \frac{(\quad +3)^n}{n4^n}$

EXAMPLE Find the interval of convergence of each series.

$(\boldsymbol{A})$

$$\sum_{k=0}^{\infty} \frac{k!\,x^{2k}}{3^k}$$

Recall

Ratio: $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| < 1$

Root: $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} < 1$

Geometric: $\sum ar^n;\ |r| < 1$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{n!}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} x^n$$

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Representing Functions as Power Series

◆ Representing complicated functions as power series lets us *approximate & manipulate* them more easily.

► To do this, we often use the form of a geometric series:

Recall

$$S = \sum_{n=0}^{\infty} a\, r^n = \frac{a}{1-r}$$

EXAMPLE

Find a power series for the function $f(x) = \frac{1}{x}$ centered at $a = 1$ and determine the interval of convergence.

HOW TO: Represent $f(x)$ as a Power Series

1) Write f in the form:

2) Identify a & r

3) Plug a & r into $\sum_{n=0}^{\infty} a\, r^n$

4) Find int. of convergence w/ *geometric series test:* $|\quad| < 1$

EXAMPLE

Find the power series representation centered at $x = 0$ of the following function. Give the interval of convergence for the resulting series.

$$f(x) = \frac{2}{2 + x}$$

HOW TO: Represent $f(x)$ as a Power Series

1) Write f in the form: $\dfrac{a}{1 - r}$

2) Identify a & r

3) Plug a & r into $\sum_{n=0}^{\infty} a\, r^n$

4) Find int. of convergence w/ *geometric series test:* $|r| < 1$

PRACTICE

Find the power series representation centered at $x = 0$ of the following function. Give the interval of convergence for the resulting series.

$$f(x) = \frac{1}{1 - x^3}$$

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Taylor Series

- Recall: A power series is a polynomial w/ infinite terms represented by

Recall

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

  - A Taylor Series is a type of power series where c_n is determined using the ______________ of a function.

EXAMPLE Find the Taylor series of $f(x) = \ln x$ centered at $a = 1$.

New **Taylor Series**

$$\sum_{n=0}^{\infty} \frac{\qquad}{\qquad}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$$

(Taylor Series centered at $x = a$)

Derivatives of f	Derivatives of f at $x = 1$
$f = \ln x$	$f(1) = \ln \quad =$
$f' =$	$f'(1) =$
$f'' =$	$f''(1) =$
$f''' =$	$f'''(1) =$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\quad)}{n!}(x - \quad)^n$$

- *Maclaurin series* is a special case of Taylor series that is centered at ____:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

(Maclaurin Series)

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PRACTICE

Find the Taylor Series of $f(x) = \cos x$ centered at $x = \pi$. Then, write the power series using summation notation.

Derivatives of f	Derivatives of f at $x = \pi$
$f =$	$f(\pi) =$
$f' =$	$f'(\pi) =$
$f'' =$	$f''(\pi) =$
$f''' =$	$f'''(\pi) =$
$f^{(4)} =$	$f^{(4)}(\pi) =$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\quad)}{n!}(x - \quad)^n$$

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EXAMPLE Answer the following questions.

($\boldsymbol{A}$) Find the Maclaurin Series of $f(x) = e^x$. Then, write the power series using summation notation.

Derivatives of $\boldsymbol{f}$	Derivatives of $\boldsymbol{f}$ at $\boldsymbol{x = 0}$
$f =$	$f(0) =$
$f' =$	$f'(0) =$
$f'' =$	$f''(0) =$
$f''' =$	$f'''(0) =$
$f^{(4)} =$	$f^{(4)}(0) =$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\quad)}{n!}(x - \quad)^n$$

($\boldsymbol{B}$) Use the Maclaurin series for $f(x) = e^x$ to find the Maclaurin series for $f(x) = e^{2x}$.

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Convergence of Taylor and Maclaurin Series

◆ Taylor & Maclaurin series have **intervals of convergence** which are found in the same way as power series.

EXAMPLE Determine the convergence of the Taylor series for $f(x) = \ln x$ centered at $x = 1$.

Write the general term: $$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n =$$

$$= 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

Use convergence test:

Recall

Ratio: $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| < 1$

Root: $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} < 1$

Geometric: $\sum ar^n; |r| < 1$

Determine endpoint convergence:

$x =$ ___: $$\sum_{n=0}^{\infty} (-1)^n \frac{(\quad -1)^{n+1}}{n+1}$$

$x =$ ___: $$\sum_{n=0}^{\infty} (-1)^n \frac{(\quad -1)^{n+1}}{n+1}$$

Write interval of convergence:

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PRACTICE Find the interval of convergence for the Taylor series for $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$.

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PRACTICE Find the interval of convergence for the Maclaurin series for $f(x) = \tan^{-1} x$.

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Taylor Polynomials

Recall
$s_n = a_1 + a_2 + a_3 + \cdots + a_n$

- A Taylor polynomial $p_n(x)$ of degree n is a ____________ of a Taylor series & is used for approximating functions.
  - Taylor polynomials centered at $x = 0$ are also called Maclaurin polynomials.

EXAMPLE Find the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = e^x$.

New **n^{th} Taylor Polynomial**

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Taylor polynomial centered at $x = a$)

$p_0(x) =$

$p_1(x) =$

$p_2(x) =$

$p_3(x) =$

EXAMPLE Approximate $e^{0.2}$ to four decimal places using the third-degree Maclaurin polynomial for $f(x) = e^x$.

$p_3(x) =$

- Using higher-________ Taylor polynomials gives more accurate approximations.

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Recall

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Taylor polynomial centered at $x = a$)

PRACTICE Answer the following questions.

(***A***) Find the Taylor polynomials of order 0, 1, 2, and 3 for $f(x) = \ln(x)$ centered at $x = 1$.

(***B***) Approximate $\ln 1.5$ to four decimal places using the third-degree Taylor polynomial for $f(x) = \ln x$.

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Recall

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Taylor polynomial centered at $x = a$)

PRACTICE Answer the following questions.

$(\boldsymbol{A})$ Find the Maclaurin polynomials of order 0, 1, 2, and 3 for $f(x) = \sin x$.

$(\boldsymbol{B})$ Approximate $\sin 0.3$ to four decimal places using the third-degree Maclaurin polynomial for $f(x) = \sin x$.

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Topic Resource: Taylor Series of Common Functions

Taylor Series (centered at $x = 0$)	Interval of Convergence
$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1,1)$
$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$	$(-1,1)$
$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^n}{n}$	$(-1,1]$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$[-1,1]$