TOPIC: INTEGRALS RESULTING IN INVERSE TRIGONOMETRIC FUNCTIONS

Integrals Resulting in Inverse Trig Functions

◆ The *derivatives* of inverse trig fcns can help us find the *integrals* of certain fcns that can't be found with other rules.

Recall

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$	$\frac{d}{dx}\sec^{-1}x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

RULES OF INTEGRATION		
Name	**Rule**	**Example**
Inverse Sine	$\int \frac{1}{\sqrt{1-x^2}}\,dx =$ ________ $+ C$	$\int \left(\frac{4}{\sqrt{1-x^2}}\right) dx =$
Inverse Tangent	$\int \frac{1}{1+x^2}\,dx =$ ________ $+ C$	$\int \left(\frac{1}{1+x^2}+\frac{1}{x^2}\right) dx =$
Inverse Secant	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx =$ ________ $+ C$	$\int \left(\frac{1}{3\lvert x\rvert\sqrt{x^2-1}}\right) dx =$

PRACTICE Evaluate the integral.

$$\int \left(\frac{2}{\sqrt{1-x^2}} - \frac{1}{3\sqrt{x}}\right) dx$$

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EXAMPLE Evaluate the integral.

$$\int_0^1 \frac{5}{1+x^2} + \frac{3}{x}\, dx$$

Recall

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$$

PRACTICE Evaluate the integral.

$$\int_{\sqrt{2}}^{2} \frac{1}{|x|\sqrt{x^2-1}}\, dx$$

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Integrals Resulting in Inverse Trig Functions: Substitution

- Using **substitution**, we get these additional integral rules that also result in inverse trig functions.
  - In these rules, $\boldsymbol{a}$ is a positive constant & $\boldsymbol{u}$ is a function of x that you differentiate to find $\boldsymbol{du}$.

Recall

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

RULES OF INTEGRATION

Name	Rule	Example
Inverse Sine	$\int \frac{1}{\sqrt{\boldsymbol{a}^2-\boldsymbol{u}^2}}\,\boldsymbol{du} = \sin^{-1}\frac{\quad}{\quad} + C$	$\int \frac{1}{\sqrt{16-9x^2}}\,dx =$
Inverse Tangent	$\int \frac{1}{\boldsymbol{a}^2+\boldsymbol{u}^2}\,\boldsymbol{du} = \frac{1}{\quad}\tan^{-1}\frac{\quad}{\quad} + C$	$\int \frac{6}{25+36x^2}\,dx =$
Inverse Secant	$\int \frac{1}{\lvert\boldsymbol{u}\rvert\sqrt{\boldsymbol{u}^2-\boldsymbol{a}^2}}\,\boldsymbol{du} = \frac{1}{\quad}\sec^{-1}\left\lvert\frac{\quad}{\quad}\right\rvert + C$	$\int \frac{7}{\lvert 7x\rvert\sqrt{49x^2-4}}\,dx =$

EXAMPLE Evaluate the integral in terms of an inverse trig function.

$$\int \frac{e^x}{4+e^{2x}}\,dx$$

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PRACTICE Find the indefinite integral.

$$\int \frac{1}{x\sqrt{4x^2 - 16}}\, dx$$

Recall

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{|u|\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

PRACTICE Evaluate the definite integral in terms of an inverse trig function.

$$\int_0^{\sqrt{3}/3} \frac{dx}{\sqrt{4 - 9x^2}}$$

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PRACTICE Evaluate the definite integral using the appropriate substitutions.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Recall

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{|u|\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

EXAMPLE Find the indefinite integral.

$$\int \frac{e^{\tan^{-1}\theta}}{1+\theta^2}\,d\theta$$

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PRACTICE Find the indefinite integral.

(***A***)

$$\int \frac{\arcsin^4 x}{\sqrt{1-x^2}}\, dx$$

(***B***)

$$\int \frac{\sec^2(\sin^{-1}\theta)}{\sqrt{1-\theta^2}}\, d\theta$$

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EXAMPLE Find the indefinite integral.

$$\int \frac{t+5}{t^2+16}\, dt$$

Recall

$$\int \frac{1}{\sqrt{a^2-u^2}}\, du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2+u^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{|u|\sqrt{u^2-a^2}}\, du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

EXAMPLE Compute the definite integral.

$$\int_{\sqrt{3}/3}^{1} \frac{2}{\arctan x\,(1+x^2)}\, dx$$

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Completing the Square to Rewrite the Integrand

◆ Recall: Using substitution, we got additional integral rules that result in inverse trig functions.

► In order to apply these rules, we may need to *complete the square* in the integrand to put in the correct form.

Recall

$x^2 + bx + c$ → $(x^2 + bx + \_\_\_\_) + (c - \_\_\_\_\_)$ → $(x + \_\_\_\_\_)^2 + \_\_\_\_\_\_$

(Completing the Square)

EXAMPLE Evaluate the given integral by rewriting it in the form of an inverse trig formula.

$$\int \frac{7}{x^2 + 6x + 13}\, dx$$

$(x^2 + 6x + \_\_\_\_) + (13 - \_\_\_\_)$

$(x + \_\_\_\_)^2 + \_\_\_\_\_$

Recall

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{|u|\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

◆ If you can't use power rule, u-sub, basic inverse trig formula, or long div. to solve an int., try completing the square.

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EXAMPLE Find the indefinite integral.

$$\int \frac{8}{\sqrt{4x - x^2}}\ dx$$

Recall

$$\int \frac{1}{\sqrt{a^2 - u^2}}\ du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{|u|\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

PRACTICE Find the indefinite integral.

$$\int \frac{-2}{x^2 - 12x + 45}\ dx$$

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EXAMPLE Match each integral $(\boldsymbol{A})$-$(\boldsymbol{E})$ to the approach you should take to evaluate it $(\mathbf{1})$-$(\mathbf{5})$.

$(\boldsymbol{A})$ $\displaystyle\int \frac{x^2 + 3x - 2}{x - 1}\, dx$

$(\mathbf{1})$ Substitution resulting in $\int \frac{du}{u}$

$(\boldsymbol{B})$ $\displaystyle\int \frac{x + 4}{x^2 + 8x + 5}\, dx$

$(\mathbf{2})$ Complete the square to use inverse trig formula

$(\boldsymbol{C})$ $\displaystyle\int \frac{2x - 3}{\sqrt{x^2 - 3x}}\, dx$

$(\mathbf{3})$ Choose u & a to use inverse trig formula

$(\boldsymbol{D})$ $\displaystyle\int \frac{5}{\sqrt{4 + 3x - x^2}}\, dx$

$(\mathbf{4})$ Long division to put in more easily integrable form

$(\boldsymbol{E})$ $\displaystyle\int \frac{4}{16x^2 + 9}\, dx$

$(\mathbf{5})$ Substitution resulting in general power rule