**TOPIC: INTEGRATION BY PARTS**

**Topic Resource: Basic Integration Rules**

| RULES OF INTEGRATION | | |
|---|---|---|
| **Basic Integration Properties** | | |
| $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ | $\int [k \cdot f(x)]\,dx = k \cdot \int f(x)\,dx$ | |
| **Basic Integration Rules** | | |
| $\int 0\,dx = C$ | $\int k\,dx = kx + C$ | |
| $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ \ n \neq -1$ | $\int f\big(g(x)\big) \cdot g'(x)\ \ dx = \int f(u)\,du$ | |
| **Integrals Resulting in Trig Functions** | | |
| $\int \cos x\,dx = \sin x + C$ | $\int \sec^2 x\ \ dx = \tan x + C$ | $\int \sec x \tan x\,dx = \sec x + C$ |
| $\int \sin x\,dx = -\cos x + C$ | $\int \csc^2 x\ \ dx = -\cot x + C$ | $\int \csc x \cot x\,dx = -\csc x + C$ |
| **Integrals of Other Trig Functions** | | |
| $\int \tan x\ \ dx = -\ln\lvert\cos x\rvert + C = \ln\lvert\sec x\rvert + C$ | $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$ | |
| $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$ | $\int \csc x\,dx = -\ln\lvert\csc x + \cot x$ | |
| **Integrals Involving Exponential & Logarithmic Functions** | | |
| $\int b^x\,dx = \frac{b^x}{\ln b} + C$ | $\int e^x\,dx = e^x + C$ | $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$ |
| $\int b^u\,du = \frac{b^u}{\ln u} + C$ | $\int e^u\,du = e^u + C$ | $\int \frac{1}{u}\,du = \ln\lvert u\rvert + C$ |
| **Integrals Resulting in Inverse Trig Functions** | | |
| $\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \sin^{-1}\frac{u}{a} + C$ | $\int \frac{1}{a^2 + u^2}\,du = \frac{u}{a}\tan^{-1}\frac{u}{a} + C$ | $\int \frac{1}{\lvert u\rvert\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\sec^{-1}\left\lvert\frac{u}{a}\right\rvert + C$ |
| **Properties of Definite Integrals** | | |
| $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ | $\int_a^a f(x)\,dx = 0$ | $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ |

## TOPIC: INTEGRATION BY PARTS

### Intro to Integration by Parts

◆ To integrate the product of two functions, if all other methods fail, use **integration by parts (IBP)**.

► To get the IBP formula, we have to __________ the derivative product rule.

**New** | **Integration by Parts**

**Recall**

$$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\frac{d}{dx}[f(x)\cdot g(x)] \quad = \quad f'(x)\cdot g(x) \quad + \quad f(x)\cdot g'(x)$$

$$\_\_\_\_\_\_\cdot\_\_\_\_\_\_ = \int \_\_\_\_\_\_\cdot\_\_\_\_\_\_\ dx + \int \_\_\_\_\_\_\cdot\_\_\_\_\_\_\ dx$$

$$\int \underbrace{\_\_\_\_\_\_}\cdot\underbrace{\_\_\_\_\_\_}\ dx = \underbrace{\_\_\_\_\_\_}\cdot\underbrace{\_\_\_\_\_\_} - \int \underbrace{\_\_\_\_\_\_}\cdot\underbrace{\_\_\_\_\_\_}\ dx$$

$$\int \boldsymbol{u}\cdot \boldsymbol{dv} = \_\_\_\cdot\_\_\_ - \int \_\_\_\cdot\_\_\_$$

◆ When using IBP, $\boldsymbol{u}$ should become simpler when *differentiated* and $\boldsymbol{dv}$ should be a function that's easily *integrated*.

**EXAMPLE** Find the indefinite integral using integration by parts.

$$\int 6xe^x\ dx$$

$\boldsymbol{u} = \_\_\_\_\_\_$ $\qquad$ $\boldsymbol{dv} = \_\_\_\_\_\_\ dx$

$\boldsymbol{du} = \_\_\_\_\_\ dx$ $\qquad$ $\boldsymbol{v} = \_\_\_\_\_\_$

◆ $\boldsymbol{u}$ will *often* be the function that appears FIRST here: **L**og, **I**nverse, **A**lgebraic, **T**rig, **E**xponential  **LIATE**

## TOPIC: INTEGRATION BY PARTS

**EXAMPLE** Find the integral.

**Recall**

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

LIATE

$(\boldsymbol{A})$

$$\int \ln x \, dx$$

---

$(\boldsymbol{B})$

$$\int \sin^{-1} t \, dt$$

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**PRACTICE** Find the integral.

$(\boldsymbol{A})$

$$\int s\, e^{3s} ds$$

---

$(\boldsymbol{B})$

$$\int \theta \cos 2\theta \, d\theta$$

---

$(\boldsymbol{C})$

$$\int x^2 \ln x \;\; dx$$

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**Repeated Integration by Parts**

◆ Integration by parts may need to be ______________ to evaluate an integral.

**EXAMPLE** Find the indefinite integral $\int 3x^2e^x\,dx$.

**Recall**

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int (3x^2e^x)\,dx = \_\_\_\_\_ \cdot \_\_\_\_\_ - \int (\_\_\_\_\_ \cdot \_\_\_\_\_)$$

$u =$ ____ $dv =$ _____

$du =$ ____ $v =$ _____

$$= \_\_\_\_\_ \cdot \_\_\_\_\_ - \left[\_\_\_\_\_ \cdot \_\_\_\_\_ - \int (\_\_\_\_\_ \cdot \_\_\_\_\_)\right]$$

$u =$ ____ $dv =$ _____

$du =$ ____ $v =$ _____

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**EXAMPLE** Find the indefinite integral.

(***A***)

$$\int (t^2 + 2t - 3)\, e^{4t}\, dt$$

---

(***B***)

$$\int (\ln x)^2\, dx$$

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**PRACTICE** Find the indefinite integral.

(***A***)

$$\int r^2 e^{-r} dr$$

---

(***B***)

$$\int \theta^2 \cos 3\theta \, d\theta$$

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**Tabular Integration by Parts**

◆ For more complicated integrals that require repeated IBP, it can be helpful to organize into a ________________.

**EXAMPLE** Find the integral $\int 3x^2 e^{2x}\,dx$.

**Recall — Integration by Parts**

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int \underbrace{(3x^2}_{u}\underbrace{e^{2x})\,dx}_{dv}$$

$$= \underbrace{3x^2}_{u}\underbrace{\left(\frac{1}{2}e^{2x}\right)}_{v} - \int \underbrace{\left(\frac{1}{2}e^{2x}}_{v} \cdot \underbrace{6x\right)\,dx}_{du}$$

$$= \underbrace{3x^2}_{u} \cdot \underbrace{\frac{1}{2}e^{2x}}_{v} - \left(\underbrace{6x}_{u} \cdot \underbrace{\frac{1}{4}e^{2x}}_{v} - \int \underbrace{\frac{1}{4}e^{2x}}_{v} \cdot \underbrace{6\,dx}_{du}\right)$$

$$= \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$$

**New — Tabular Integration by Parts**

$u$ & its **D**erivatives (D column); $dv$ & its antiderivatives (**I**ntegrals) (I column)

| | Alternating Signs | $D$ | $I$ |
|---|---|---|---|
| Start with: | [ + \| − ] | | |
| | [ + \| − ] | | |
| | [ + \| − ] | | |
| | [ + \| − ] | | |

$\int (3x^2e^{2x})\,dx = (\_\_\_\_)(\_\_\_\_\_)\quad (\_\_\_\_)(\_\_\_\_\_)$

$(\_\_\_\_)(\_\_\_\_)\quad \int (\_\_\_\_)(\_\_\_\_) + C$

$=$

◆ In the $D$ column, differentiate $u$ to ______ if possible or until the product of the bottom row is a basic integral.

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**EXAMPLE** Evaluate the indefinite integral using the tabular method.

(***A***)

$$\int t^3 \ln t \, dt$$

---

(***B***)

$$\int e^t \sin 3t \, dt$$

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**PRACTICE** Evaluate the indefinite integral.

**(A)**

$$\int (s^2 + 4)e^{3s}\, ds$$

---

**(B)**

$$\int \frac{\ln x}{x^4}\, dx$$

---

**(C)**

$$\int x^2 \cos 4x \; dx$$

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**Integration by Parts: Definite Integrals**

◆ To find a definite integral using IBP, solve like an indefinite integral, then apply bounds to __________ parts at the end.

**EXAMPLE** Evaluate the definite integral using integration by parts.

$$\int_0^2 6xe^x \, dx$$

New

$$\int_{\_\_}^{\_\_} u \cdot dv = u \cdot v \Big|_{\_\_}^{\_\_} - \int_{\_\_}^{\_\_} v \cdot du$$

**TOPIC: INTEGRATION BY PARTS**

**PRACTICE** Evaluate the definite integral.

**Recall**

$$\int_a^b u \cdot dv = u \cdot v \Big|_a^b - \int_a^b v \cdot du$$

$(\boldsymbol{A})$

$$\int_1^2 x \ln x \;\, dx$$

---

$(\boldsymbol{B})$

$$\int_0^{\pi} t^2 \sin t \;\, dt$$

## TOPIC: INTEGRATION BY PARTS

**EXAMPLE** Find the area of the region bounded by $f(x) = (x-1)e^{3x}$ and the $x$-axis on $[1,4]$.

**Recall**

$$A = \int_a^b [f(x) - g(x)]dx$$

*(Area Between Curves)*

**EXAMPLE** Find the volume of the solid generated when the region bounded by $y = \sqrt{x \cos x}$, the $x$-axis, and the lines $x = 0$ and $x = \pi/2$ are revolved about the $x$-axis.

**Recall**

$$Volume = \int_a^b \pi[R(x)]^2 dx$$

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**PRACTICE**

The rate of growth of a particular bacteria is given by $b'(t) = 4t^2 e^{.5t}$ where $t$ is time in days. What is the total growth of the population of bacteria during the first 5 days?

**Recall**

$$\int_a^b u \cdot dv = u \cdot v \Big|_a^b - \int_a^b v \cdot du$$