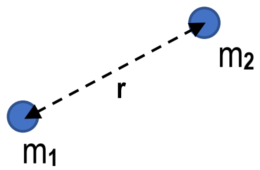
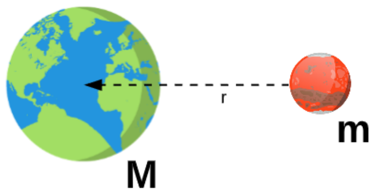
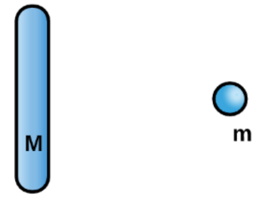
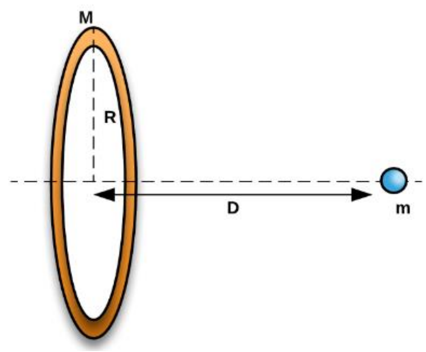
CONCEPT: Mass Distributions with Calculus

● For non-spherical distributions of mass, we use calculus to calculate the Gravitational Force.

Point Masses	Spherical Masses	Non-Spherical Mass Distributions
		
$F_g = G\frac{m_1 m_2}{r^2}$	$F_g = G\frac{Mm}{r^2}$	- Each tiny mass **dM** produces a tiny force **dF** = ________ - To solve, add up each tiny force **dF** → ______________ $F_g =$ _______ = ________ = ____________

EXAMPLE: Calculate the gravitational force on a mass **m** at a distance **D** along the axis passing through the center of a ring of mass **M** and radius **R.**



MASS DISTRIBUTIONS W/ CALC
1) Write $\mathbf{F} = \int \mathbf{dF} = \mathbf{Gm}\int \frac{\mathbf{dM}}{\mathbf{r^2}}$
2) Pick 2 **dM**'s, write an expression for **r** using the problem's geometry (save this for later!)
3) Break $\int \mathbf{dF}$ into $\int \mathbf{dF}_x$ & $\int \mathbf{dF}_y$, cancel symmetrically opposite components
4) Expand $\mathbf{F_x}$ or $\mathbf{F_y}$ into sin & cos; rewrite sin(θ) or cos(θ) in terms of the side lengths given
5) Plug in expression for **r** from Step 2, then pull all constants out of the integral
6a) If you're only left with $\int$ **dM,** replace $\int$ **dM** → **M**, and you're done!
6b) Otherwise, change differential **dM** to match the *changing* variable of integration using **density:** $\mathbf{dM = \lambda dx = \frac{M}{L}dx \quad = \sigma dA = \frac{M}{A}dA \quad = \rho dV = \frac{M}{V}dV}$
7) Determine limits of integration
8) Use integration techniques or tables to integrate

EXAMPLE: **a)** Set up the integral expression for the gravitational force between a rod of mass **M** and length **L** and a mass **m** at an arbitrary distance **D** from the end of the rod. **b)** Evaluate the integral.

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6b) Otherwise, change differential **dM** to match the *changing* variable of integration using **density:**

$$\mathbf{dM} = \boldsymbol{\lambda}\mathbf{dx} = \frac{\mathbf{M}}{\mathbf{L}}\mathbf{dx} \quad = \boldsymbol{\sigma}\mathbf{dA} = \frac{\mathbf{M}}{\mathbf{A}}\mathbf{dA} \quad = \boldsymbol{\rho}\mathbf{dV} = \frac{\mathbf{M}}{\mathbf{V}}\mathbf{dV}$$

7) Determine limits of integration

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PRACTICE: **a)** Set up the integral for the gravitational force between a rod with a density **λ** and length **2L**, and a mass **m** at an arbitrary distance **D** directly above the midpoint of the rod. b) Evaluate the integral (*Hint:* $\int \frac{dx}{(x^2+D^2)^{3/2}} = \frac{x}{D^2\sqrt{x^2+D^2}}$)

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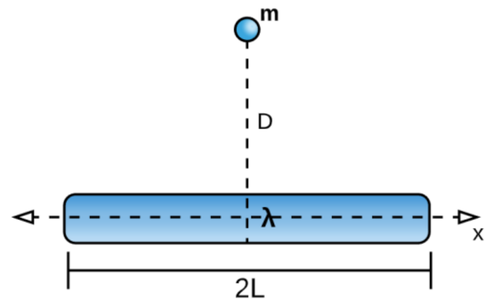
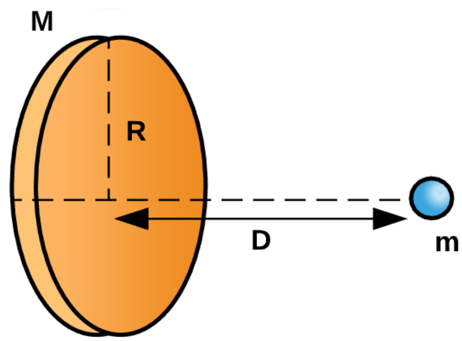


EXAMPLE: What is the gravitational force between a solid disk of mass **M** and radius **R**, and a mass **m** located a distance **D** measured along the axis passing through the center of the disk? (*Hint:* Think of the disk as being made up of many very thin concentric rings of radius **r'**, and add up all the rings.)



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